Linear independence

Consider the vectors $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ in $\mathbb{R}^2$.

The span of these vectors is $\mathbb{R}^2$, but notice 2 things:

1.) $\operatorname{Span}\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\} = \mathbb{R}^2$ as well.

2.) The vector $\begin{bmatrix}a\\b\end{bmatrix}$ can be written in many different ways as a linear combination of these 3 vectors:

$$\begin{bmatrix}a\\b\end{bmatrix} = a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = (a-b)\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}1\\1\end{bmatrix}$$

$$= 1\begin{bmatrix}1\\0\end{bmatrix} + (1-a+b)\begin{bmatrix}0\\1\end{bmatrix} + (a-1)\begin{bmatrix}1\\1\end{bmatrix}, \text{ etc.}$$

Choosing a linearly independent set of vectors ensures that neither of these things can occur:

Def: A set of vectors $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is <u>linearly independent</u> if it satisfies the following:

If $t_1\vec{x}_1 + t_2\vec{x}_2 + \dots + t_k\vec{x}_k = 0$, then $t_1 = t_2 = \dots = t_k = 0$.

Otherwise it is <u>linearly dependent</u>.

Theorem: If $\{\vec{x}_1, \dots, \vec{x}_k\}$ is linearly independent, then every vector in $\operatorname{span}\{\vec{x}_1, \dots, \vec{x}_k\}$ has a <u>unique</u> representation as a linear combination of the $\vec{x}_i$.

Why? If $\vec{v} = a_1\vec{x}_1 + \dots + a_k\vec{x}_k$ and
$\vec{v} = b_1\vec{x}_1 + \dots + b_k\vec{x}_k$, then

$$a_1\vec{x}_1 + \dots + a_k\vec{x}_k = b_1\vec{x}_1 + \dots + b_k\vec{x}_k$$

$$\Rightarrow (a_1-b_1)\vec{x}_1 + \dots + (a_k-b_k)\vec{x}_k = 0.$$

Thus, since they are linearly independent,

$a_i - b_i = 0 \Rightarrow a_i = b_i$ for each i, so there is only one way to write $\vec{v}$ as a linear combination of $\vec{x}_1, \dots, \vec{x}_n$.

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$ linearly independent?

Assume $a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \vec{0}$.

We want to see if a, b, c must be 0. This becomes a system of equations:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

So $a = b = c = 0$ is the only solution. So they are linearly independent.

Ex: Are $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ linearly indep?

If $a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}1\\-1\end{bmatrix} + c\begin{bmatrix}0\\1\end{bmatrix} = \vec{0}$, then we have

$\left[\begin{array}{ccc|c}1 & 1 & 0 & 0\\1 & -1 & 1 & 0\end{array}\right]$. Since this is homogeneous in 3 variables, and there are just 2 equations, there must be infinitely many (and thus nontrivial) solutions. Thus, they are not linearly independent.

We summarize this strategy as follows:

Linear independence test:

To check if $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent,

1.) Set up a homog. system of equations

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \ldots + t_k\vec{x}_k = \vec{0}, \text{ i.e.}$$

$$\left[\begin{array}{cccc|c}\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_k & \vec{0}\end{array}\right]$$

2.) If there is a nontrivial solution (i.e. any parameter), they are not lin. indep., otherwise they are.

Ex: The standard basis is linearly indep.:

$\left[\begin{array}{cccc|c}\vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_k & \vec{0}\end{array}\right] = \left[\begin{array}{c|c}I & \vec{0}\end{array}\right]$, which has only the trivial solution.

Ex: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ linearly independent?

If

$$a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \vec{0}$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Only 2 leading ones, so there are inf. many solutions, so it's not indep.

Ex: Can the set $\{\vec{0}, \vec{x}_1, \dots, \vec{x}_k\}$ be linearly independent?

No: $\underbrace{1}_{\text{nonzero coefficient}}(\vec{0}) + 0\vec{x}_1 + \dots + 0\vec{x}_k = 0$

No set containing $\vec{0}$ is ever linearly independent.

Ex: If $\{\vec{x}, \vec{y}\}$ is linearly independent, is $\{\vec{x}+\vec{y}, \vec{x}-\vec{y}\}$?

If $a(\vec{x}+\vec{y}) + b(\vec{x}-\vec{y}) = 0$, then

$(a+b)\vec{x} + (a-b)\vec{y} = 0$, so

$a+b=0, \quad a-b=0 \Rightarrow a=0=b$, so

$\{\vec{x}+\vec{y}, \vec{x}-\vec{y}\}$ is also lin. independent.

Ex: If $\{\vec{v}, \vec{w}\}$ is linearly dependent, and both $\vec{v}$ and $\vec{w}$ are nonzero, then

$$s\vec{v} + t\vec{w} = \vec{0} \text{ for some } s,t \neq 0.$$

$$\Rightarrow s\vec{v} = -t\vec{w}, \text{ so } \vec{v} \text{ and } \vec{w} \text{ must be parallel.}$$

Conversely, if we assume $\vec{v}$ and $\vec{w}$ are parallel vectors then $\vec{v} = k\vec{w}$, some $k \neq 0$, so

$$\vec{v} - k\vec{w} = \vec{0}, \text{ so } \{\vec{v}, \vec{w}\} \text{ is linearly dependent.}$$

Thus, $\{\vec{v}, \vec{w}\}$ is lin. dependent if and only if $\vec{v}$ and $\vec{w}$ are parallel.

What about 3 vectors?

Ex: If $\{\vec{u}, \vec{v}, \vec{w}\}$ is dependent (in $\mathbb{R}^3$), then

$$a\vec{u} + b\vec{v} + c\vec{w} = 0, \quad \text{where } a, b, \text{or } c \text{ is nonzero.}$$

Assume $a \neq 0$.

Then $a\vec{u} = -b\vec{v} - c\vec{w} \Rightarrow \vec{u} = \frac{-b}{a}\vec{v} - \frac{c}{a}\vec{w}$, so

$\vec{u}$ is in span$\{\vec{v}, \vec{w}\}$, which is a plane or a line.

Thus, all 3 vectors must lie in the same plane.

The converse holds as well. So $\{\vec{u}, \vec{v}, \vec{w}\}$ is lin. dependent if and only if $\vec{u}, \vec{v}, \vec{w}$ all lie in the same plane. i.e. if span$\{\vec{u}, \vec{v}, \vec{w}\}$ is a plane or a line.

Matrices + linear independence

A an $m \times n$ matrix with columns $\vec{c}_1, \dots, \vec{c}_n$.

If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ then

$$A\vec{x} = [\vec{c}_1 \cdots \vec{c}_n]\,\vec{x} = x_1\vec{c}_1 + x_2\vec{c}_2 + \dots + x_n\vec{c}_n.$$

If we set $A\vec{x} = \vec{0}$, there are nontrivial solutions $\Leftrightarrow$ $\vec{c}_1, \dots, \vec{c}_n$ are linearly dependent.

If instead we set $A\vec{x} = \vec{b}$, then there is a solution if and only if $\vec{b} = x_1\vec{c}_1 + \dots + x_n\vec{c}_n$ for some $x_1, \dots, x_n$, i.e. $\vec{b}$ is in span$\{\vec{c}_1, \dots, \vec{c}_n\}$.

We can summarize these in the following theorem:

Theorem: If $A = [\vec{c}_1 \cdots \vec{c}_n]$ is an $m \times n$ matrix, then

1.) $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ is linearly independent if and only if $A\vec{x} = \vec{0}$ has no nontrivial solutions.

2.) $\mathbb{R}^m = \text{span}\{\vec{c}_1, \ldots, \vec{c}_n\}$ if and only if $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$.

For square matrices, we can relate these conditions to invertibility:

Theorem: If A is an $n \times n$ matrix, the following are equivalent:

1.) A is invertible.

2.) The columns of A are linearly independent.

3.) The columns of A span $\mathbb{R}^n$.

4.) The rows of A are linearly independent.

5.) The rows of A span $\mathbb{R}^n$.

Now we have a simpler way to check linear indep. if we have n vectors in $\mathbb{R}^n$:

Ex: Are $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ linearly indep?

Set $A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$.

$$\det A = -1 \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + 1 \begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= -1(-1+2) + 1(0-6)$$

$$= -1(1) + 1(-6) = -1-6 = -7$$

so A is invertible, so they are linearly independent, and they span $\mathbb{R}^3$.

Dimension

Very important theorem: If U is a subspace of $\mathbb{R}^n$, and U is spanned by m vectors, then if U contains k linearly independent vectors, then $k \leq m$.

A maximum collection of linearly indep. vectors in a subspace is called a basis. More precisely:

Def: If U is a subspace of $\mathbb{R}^n$, a set of vectors $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ in U is a basis if:

1.) $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$ is linearly independent, and

2.) $U = \text{span}\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_m\}$

It turns out that any two bases for a subspace will have the same # of vectors.

i.e. if $\{\vec{y}_1, \ldots, \vec{y}_k\}$ is another basis for U, then $\text{span}\{\vec{y}_1, \ldots, \vec{y}_k\} = U$ and $\{\vec{x}_1, \ldots, \vec{x}_m\}$ is a set of linearly independent vectors in U, so $m \le k$. By the same argument, $k \le m$. Thus $m = k$.

This number is called the dimension of U:

Def: If U is a subspace of $\mathbb{R}^n$ and $\{\vec{x}_1, \ldots, \vec{x}_m\}$ is a basis of U, then the <u>dimension</u> of U is m, denoted $\dim U = m$.

Ex: If $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is the standard basis in $\mathbb{R}^n$, then $\text{span}\{\vec{e}_1, \ldots, \vec{e}_n\} = \mathbb{R}^n$ and it's linearly independent, so $\{\vec{e}_1, \ldots, \vec{e}_n\}$ <u>is</u> indeed a basis, and $\dim \mathbb{R}^n = n$.

Ex: We saw above that any invertible $n \times n$ matrix has lin. indep rows (and cols) that span $\mathbb{R}^n$. Thus, the rows (or cols) of any invertible matrix forms a basis for $\mathbb{R}^n$.

Note: A key takeaway here is that there is more than one possible <u>basis</u> for each subspace, but the dimension will always be the same.

Ex: Let $U = \left\{ \begin{bmatrix} s \\ s+t \\ s \end{bmatrix} \middle| \ s, t \text{ in } \mathbb{R} \right\}$.

Is this a subspace of $\mathbb{R}^3$?

Notice that $\begin{bmatrix} s \\ s+t \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

so since s and t can be any real #s,

$U = \text{span}\left\{ \underset{\vec{u}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \underset{\vec{v}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} \right\}$, so it is a subspace, since the span of any set of vectors is a subspace.

What is $\dim U$? (We know $\dim U \leq 2$. Why? By the very important theorem. Any basis will be lin. indep so the number of basis elts can be at most the # of spanning vectors.)

$\{\vec{u}, \vec{v}\}$ spans U so we need to check if it is lin. indep.

Since neither $\vec{u}$ nor $\vec{v}$ is a scalar multiple of the other, they are linearly independent. Thus $\{\vec{u}, \vec{v}\}$ is a basis, so $\dim U = 2$.

Ex: $\dim \{0\} = 0$. There is no linearly independent collection of vectors in $\{0\}$.

Theorem: Let $U \neq \{0\}$ be a subspace of $\mathbb{R}^n$. Then:

1.) U has a basis and $\dim U \leq n$.

2.) Any linearly independent set in U can be enlarged to a basis of U (by adding vectors from a fixed basis of U).

3.) Any spanning set for U can be cut down to a basis of U (by deleting vectors).

Ex: If $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, $\{\vec{u}, \vec{v}\}$ is linearly independent in $\mathbb{R}^3$. Thus, we should be able to add a vector from the standard basis to $\{\vec{u}, \vec{v}\}$ to make it a basis.

Does $\vec{e}_1$ work? i.e. is $\{\vec{u}, \vec{v}, \vec{e}_1\}$ linearly independent and span $\mathbb{R}^3$? If

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \text{then} \quad \det A = -1, \text{ so}$$

A is invertible, so $\{\vec{u}, \vec{v}, \vec{e}_1\}$ is a basis.

In general, if $\dim U = m$ and we have m vectors which either span U <u>or</u> are lin. independent, then they form a basis. That is:

Theorem: If U is a subspace of $\mathbb{R}^n$ where $\dim U = m$, let $B = \{\vec{x}_1, \dots, \vec{x}_m\}$, m vectors in U. Then

B is linearly independent if and only if $\operatorname{span} B = U$.

Ex: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ spans $\mathbb{R}^2$,

and any pair forms a basis, since each pair is lin. indep. So, e.g. $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^n$.

If one subspace is inside of another, how do their dimensions compare?

Theorem: If U and V are subspaces of $\mathbb{R}^n$, and $U \subseteq V$, i.e. U is contained in V, then

1.) $\dim U \leq \dim V$.

2.) If $\dim U = \dim V$, then $U = V$.

Ex: If U is a subspace of $\mathbb{R}^3$, then

$\dim U = 3 \iff U = \mathbb{R}^3$,

$\dim U = 2 \iff U$ is a plane through the origin,

$\dim U = 1 \iff U$ is a line through the origin.

Ex: (3a) Let $U = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 9 \\ -6 \\ 6 \end{bmatrix} \right\}$.

What is $\dim U$? We know that $\dim U \le 3$, and some subset of these 3 vectors form a basis, so we first check if they are lin. independent:

$$a \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix} + c \begin{bmatrix} 1 \\ 9 \\ -6 \\ 6 \end{bmatrix} = \vec{0}$$

$$\Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 3 & 9 & 0 \\ 2 & 0 & -6 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 10 & 0 \\ 0 & -4 & -8 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ $\Rightarrow$ there's a parameter, so

they are <u>not</u> lin. independent.

However, $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \\ 3 \end{bmatrix} \right\}$ <u>is</u> lin. independent,

so it forms a basis for U, so $\dim U = 2$.

Practice problems: 5.2 : 1, 2bd, 3bcd, 4cdf, 6adf